Fundamental Knowledge on Nature, Physics, Geometry and Mathematics

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Editor & Publisher

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All feedback will be used to improve the next edition. Any people who want to translate it in Nepali language also welcomed warmly.

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Curiosity on Nature

Change is the most fundamental observation about nature at large. It turns out that everything that happens in the world is certain types of changes. There are no exceptions. Change is also important to the human condition. It is simply the part of human experience.

Change Categories

Three categories of change are commonly recognized:

Growth

This category of change, is observed for animals, plants, bacteria, crystals, mountains, planets, stars and even galaxies. In the nineteenth century, changes in the population of systems, biological evolution, and in the twentieth century, changes in the size of the universe, cosmic evolution, were added to this category. Traditionally, these phenomena were studied by separate sciences. Independently they all arrived at the conclusion that growth is a combination of material transport and transformation.

Material Transport

The only type of change we call movement in everyday life is material transport, such as a person walking, a leaf falling from a tree, or a musical instrument playing. Transport is the change of position or orientation of objects, fluids included.

Early scholars differentiated types of transport by their origin. Movements such as those of the legs when walking were classified as volitional, because they are controlled by one's will, whereas movements of external objects, such as the fall of a snowflake, which cannot be influenced by will-power, were classified as passive.

The way machines work forced scholars to rethink the distinction between volitional and passive motion. Like living beings, machines are self-moving and thus mimic volitional motion. However, careful observation shows that every part in a machine is moved by another, so their motion is in fact passive.

Transformation

Another category of change groups such as the dissolution of salt in water, the formation of ice by freezing, the rotting of wood, the cooking of food, the coagulation of blood, and the melting and alloying of metals.

These changes of color, brightness, hardness, temperature and other material properties are all transformations. Transformations are changes not visibly connected with transport. To this category, a few ancient thinkers added the emission and absorption of light. In the twentieth century, these two effects were proven to be special cases of transformations, as were the newly discovered appearance and disappearance of matter, as observed in the Sun and in radioactivity. Mind change, such as change of mood a type of transformation.

Key Point

Every type of change is due to the motion of particles or physical object as whole.

Perception

Human beings enjoy perceiving. Perception starts before birth, and we continue enjoying it.

Perception is first of all the ability to distinguish. We use the basic mental act of distinguishing in almost every instant of life; for example, during childhood we first learned to distinguish familiar from unfamiliar observations. This is possible in combination with another basic ability, namely the capacity to memorize experiences. Memory gives us the ability to experience, to talk and thus to explore nature. Perceiving, classifying and memorizing together form learning. Without any one of these three abilities, we could not study anything.

Children rapidly learn to distinguish permanence from variability. They learn to recognize human faces, even though a face never looks exactly the same each time it is seen. From recognition of faces, children extend recognition to all other observations. Recognition works pretty well in everyday life; it is nice to recognize friends, even at night. The act of recognition thus always uses a form of generalization. When we observe, we always have some general idea in our mind.

In order to be able to define change or motion properly, an abstraction of the idea concerning the physical object and the environment is required, and to distinguish them from each other.

What distinguishes the physical object with the environment? In everyday life we would say: the situation or configuration of the involved entities. The situation somehow describes all those aspects that can differ from case to case. It is customary to call the list of all variable aspects of a set of objects their (physical) state of motion, or simply their state..

Two similar objects can differ, at each instant of time, in their — position,— velocity,— orientation. These properties determine the state and pinpoint the individuality of a physical system among exact copies of itself. Equivalently, the state describes the relation of an object or a system with respect to its environment. Or, again, equivalently:

Key Point

The state describes all aspects of a system that depend on the observer.

In addition, physical objects are distinguished by their permanent, intrinsic properties. Some examples are

- Shape

- Color
- Composition
- Mass

Intrinsic properties do not depend on the observer. They are permanent – at least for a certain time interval. Intrinsic properties also allow to distinguish physical systems from each other. And again, we can ask: What is the complete list of intrinsic properties in nature?

Describing nature as a collection of permanent entities and changing states is the starting point of the study of motion. Every observation of motion requires the distinction of permanent, intrinsic properties – describing the objects that move – and changing states – describing the way the objects move. Using the terms just introduced, we can say

Key Point

Motion is the change of state of permanent objects.

Motion to Continuity of Time

The simplest description of motion is the one we all use throughout our everyday life:

Only one thing can be at a given spot at a given time.

This general description can be separated into three assumptions: matter is impenetrable and moves, time is made of instants, and space is made of points. Without these three assumptions (do you agree with them?) it is not possible to define motion. We thus need points embedded in continuous space and time to talk about motion.

When we throw a stone through the air, we can define a sequence of observations. Our memory and our senses give us this ability. The sense of hearing registers the various sounds during the rise, the fall and the landing of the stone. Our eyes track the location of the stone from one point to the next. All observations have their place in a sequence, with some observations preceding them, some observations simultaneous to them, and still others succeeding them. We say that observations are perceived to happen at various instants – also called 'points in time' – and we call the sequence of all instants time. An observation that is considered the smallest part of a sequence, i.e., not itself a sequence, is called an event. Events are central to the definition of time.

Sequential phenomena have an additional property known as duration. Duration expresses the idea that sequences take some interval of time. We say that a sequence takes time to express that other sequences can take place in parallel with it. Beginning at a very young age, we develop the concept of 'time' from the comparison of motions in their surroundings. Grown-ups take as a standard the motion of the Sun and call the resulting type of time local time. From the Moon they deduce festival time.

Time is a concept necessary to distinguish between observations. In any sequence of observations, we observe that events succeed each other smoothly, apparently without end. In this context, 'smoothly' means that observations that are not too distant tend to be not too different. Yet between two instants, as close as we can observe them, there is always room for other events.

Duration, or time intervals, measured by different people with different clocks agree in everyday life; moreover, all observers agree on the order of a sequence of events. Time is thus unique in everyday life. In particular, all animal brains have internal clocks. These brain clocks allow their users to distinguish between present, recent and past data and observations.

Time is not only an aspect of observations, it is also a facet of personal experience. Even in our innermost private life, in our thoughts, feelings and dreams, we experience sequences and duration. Children learn to relate this internal experience of time with external observations, and to make use of the sequential property of events in their actions. Studies of the origin of psychological time show that it coincides – apart from its lack of accuracy – with clock time. Every living human necessarily uses in his daily life the concept of time as a combination of sequence and duration; this fact has been checked in numerous investigations. The term 'when' exists in all human languages.

Key Point

All methods for the definition of time are thus based on comparisons of motions. Negating all physical object and motion, it is difficult to accept separate existence of time.

Time does not exist in itself, but only through the perceived objects, from which the concepts of past, of present and of future ensue. The expression 'the flow of time' is often used to convey that in nature change follows after change, in a steady and continuous manner. Time is a concept introduced specially to describe the flow of events around us; it does not itself flow, it describes flow.

Approximate Values of Some Measured Time Intervals	Time Interval (s)
Shortest measurable time	5.39×10^{-44}
One year	3.2×10^{7}
Average One day	8.6×10^{4}
Average day 400 million years ago	7.92×10^{4}
Human 'instant'	2×10^{-4}
Age of the Universe	5×10^{17}
Age of the Earth	1.3×10^{17}
Average age of a college student	6.3×10^{8}
One class period	3.0×10^{3}
Period of pulsar (rotating neutron star) PSR 1913+16	0.059029995271(2)
Time interval between normal heartbeats	8×10^{-1}
Period of audible sound waves	10^{-3}
Period of typical radio waves	10^{-6}
Period of vibration of an atom in a solid	10^{-13}
Period of visible light waves	10^{-15}
Duration of a nuclear collision	10^{-22}
Time interval for light to cross a proton	10^{-24}

Space

Whenever we distinguish two objects from each other, we first of all distinguish their positions. We distinguish positions with our senses of sight, touch and hearing. Position is therefore an important aspect of the physical state of an object. A position is taken by only one object at a time. Positions are limited. The set of all available positions, called (physical) space, acts as both a container and a background.

During childhood, humans learn to bring together the various perceptions of space, namely the visual, the tactile, the auditory, etc., into one self consistent set of experiences and description. The result of this learning process is a certain concept of space in the brain. Indeed, the question 'where?' can be asked and answered in all languages of the world.

Closely related to space and position is size, the set of positions an object occupies. Small objects occupy only subsets of the positions occupied by large ones. In particular, objects can take positions in an apparently continuous manner: there indeed are more positions than can be counted. Size is captured by defining the distance between various positions, called length, or by using the field of view an object takes when touched, called its surface area. Length and area can be measured with the help of a meter stick. Meter sticks work well only if they are straight. But when humans lived in the jungle, there were no straight objects around them. The length of objects is independent of the person measuring it, of the position of the objects and of their orientation. Being more precise, adults derive space from distance measurements. The concepts of length, area, volume, angle and solid angle are all deduced with their help. Geo-meters, surveyors, architects, astronomers, carpet salesmen and producers of meter sticks base their trade on distance measurements.

Key Point

Space is formed from all the position and distance relations between objects.

Like time intervals, length intervals can be described most precisely with the help of real numbers. In order to simplify communication, standard units are used, so that everybody uses the same numbers for the same length. Units allow us to explore the general properties of space experimentally: space (the container of objects) is isotropic (every direction is same), homogeneous (every part is same), three-dimensional, continuous and unbounded, flat and unique or absolute (same for every observer).

There are no limits to distance, length and thus to space. Experience shows us that space has three dimensions; we can define sequences of positions in precisely three independent ways. Indeed, the inner ear of (practically) all vertebrates has three semicircular canals that sense the body's acceleration in the three dimensions of space. Similarly, each human eye is moved by three pairs of muscles.

Having defined space and time, we can precisely define motion as **change of position in space with time.**

Stating that '**motion is the change of position with time**' is neither an explanation nor a definition, since both the concepts of time and position are deduced from motion itself. It is only a description of motion. But what about the question concerning to the physical object?

Approximate Values of Some Measured Lengths	Length (m)
The shortest length	1.62×10^{-35}
3 Space dimensions up to	10^{26}
Human hair	30 to 80 $\times 10^{-6}$
Total length of DNA in each human cell	2
Distance from the Earth to the most remote known quasar	1.4×10^{26}
Distance from the Earth to the most remote normal galaxies	9×10^{25}
Distance from the Earth to the nearest large galaxy (Andromeda)	2×10^{22}
Distance from the Sun to the nearest star (Proxima Centauri)	4×10^{16}
One light-year	9.46×10^{15}
Mean orbit radius of the Earth about the Sun	1.50×10^{11}
Mean distance from the Earth to the Moon	3.84×10^{8}
Distance from the equator to the North Pole	1.00×10^{7}
Mean radius of the Earth	6.37×10^{6}
Typical altitude (above the surface) of a satellite orbiting the Earth	2×10^{5}
Length of a housefly	5×10^{-3}
Size of smallest dust particles,	10^{-4}
Size of cells of most living organisms,	10^{-5}
Diameter of a hydrogen atom,	10^{-10}
Diameter of an atomic nucleus,	10^{-14}
Diameter of a proton	10^{-14}

2

Quantities

Most scientific laws are based on experimental observations and quantitative measurements. The main objectives of physics are to identify a limited number of fundamental laws that govern natural phenomena and use them to develop theories that can predict the results of future experiments.

Physical Quantities

A physical quantity is needed to describe a property of any physical object quantitatively, i.e. with a numerical value.

We look at the statement m = 5kg

Here is m a physical quantity called mass, 5 a numerical value and kg a measuring unit. We try to classify physical quantities, and this by looking at the geometrical entity they refer to:

- **Value refers to a point:** velocity, temperature, pressure, electric potential, density,...

- **Value refers to a surface area:** all currents: momentum current, electric current, energy current,...
- **Value refers to a region of space:** mass, momentum, electric charge, entropy, energy,..

If a quantity refers to a point, its value can change from point to point. This is evident for temperature and for pressure. Maybe it is not so obvious that also the velocity is part of this category. This is because all points of a moving body have the same velocity, or don't they? It is sufficient to look at a rotating body to convince yourself of the opposite. In such a body, each point has a different velocity. Also the velocity of water in a river changes from height to height. Quantities that refer to a region of space are called substance-like quantities. Not all quantities fulfill this pattern, e.g. time, but also the spring constant, electric resistance and capacitance.

Scalars and Vectors

Let us look once again at the different physical quantities but this time from another perspective. We will compare two specifications at first: a temperature and a velocity. You can imagine it to be the air temperature and the wind velocity at a well-defined place and at a well-defined instant of time.

The specifications are t = 19 °C and v = 5 m/s.

Do you notice that one of the two specifications is incomplete? Although we know how fast the air is moving, i.e. 5 m/s, we do not know yet in which direction it is moving. Things are clear for the temperature as the temperature has no direction. Quantities such as the temperature that are defined by a single number are called scalars. Quantities for which a direction has to be specified in addition are called vectors. Here some examples:

- **Scalars:** energy, mass, electric charge, electric current strength, temperature, entropy
- **Vectors:** velocity, momentum, momentum current (also called force)

The velocity is (1) a quantity that refers to a point and (2) a vector.

This means:

- Its value can be different from point to point; it forms a distribution.
- In each point it has a well-defined direction.

Units and Measurements

In 1960, an international committee established a set of standards for the fundamental quantities of science. It is called the SI (Système International), and its fundamental units of length, mass, and time are the meter, kilogram, and second, respectively. Other standards for SI fundamental units established by the committee are those for temperature (the kelvin), electric current (the ampere), luminous intensity (the candela), and the amount of substance (the mole).

- Before 1967, the standard of time was defined in terms of the mean solar day. This definition is based on the rotation of one planet, the Earth. Therefore, this motion does not provide a time standard that is universal. In 1967, the second was redefined to take advantage of the high precision attainable in a device known as an atomic clock, which measures vibrations of cesium atoms. One second is now defined as 9 192 631 770 times the period of vibration of radiation from the cesium-133 atom.

- As recently as 1960, the length of the meter was defined as the distance between two lines on a specific platinum–iridium bar stored under controlled conditions in France. Current requirements of science and technology, however, necessitate more accuracy than that with which the separation between the lines on the bar can be determined. In the 1960s and 1970s, the meter was defined as 1 650 763.73 wavelengths1 of orange-red light emitted from a krypton-86 lamp. In October 1983, however, the meter was redefined as the distance traveled by light in vacuum during a time of 1/299 792 458 second. In effect, this latest definition establishes that the speed of light in vacuum is precisely 299 792 458 meters per second.
- The SI fundamental unit of mass, the kilogram (kg), is defined as the mass of a specific platinum–iridium alloy cylinder kept at the International Bureau of Weights and Measures at Sèvres, France. This mass standard was established in 1887 and has not been changed since that time because platinum–iridium is an unusually stable alloy.
- The ampere, symbol A, is the SI unit of electric current. It is defined by taking the fixed numerical value of the elementary charge e to be $1.602\ 176\ 634 \times 10^{-19}$ when expressed in the unit C, which is equal to A · s.' (2019).

- The kelvin, symbol K, is the SI unit of thermodynamic temperature. It is defined by taking the fixed numerical value of the Boltzmann constant k to be 1.380649×10^{-23} when expressed in the unit $J \cdot K^{-1}$.' (2019)
- The mole, symbol mol, is the SI unit of amount of substance. One mole contains exactly $6.02214076 \times 10^{23}$ elementary entities.' (2019)
- The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and has a radiant intensity in that direction of (1/683) watt per steradian.'

We note that both time and length units are defined as certain properties of a standard example of motion, namely light.

All SI units are made more practical by the introduction of standard names and abbreviations for the powers of ten, the so-called prefixes.

Power	Power	Power	Power
10^{1} deca da	10^{-1} deci d	10^{18} Exa E	10^{-18} atto a
10^{2} hecto h	10^{-2} centi c	10^{21} Zetta Z	10^{-21} zepto z
10^{3} kilo k	10^{-3} milli m	10^{24} Yotta Y	10^{-24} yocto y
10^{6} Mega M	10^{-6} micro µ	10^{27} Xenta X	10^{-27} xenno x
10^{9} Giga G	10^{-9} nano n	10^{30} Wekta W	10^{-30} weko w
10^{12} Tera T	10^{-12} pico p	10^{33} Vendekta V	10^{-33} vendeko v
10^{15} Peta P	10^{-15} femto f	10^{36} Udekta U	10^{-36} udeko u

3

Study of Shape

Combining the basic terms point, line, notion of area, plane and space, we can describe shape and size of physical object and relate them with other physical properties too. The shape and size is the most fundamental property of every object.

Study of shape combines visual delights and powerful abstractions, concrete intuitions and general theories, historical perspective and contemporary applications, and surprising insights and satisfying certainty.

Spatial thinking fuses reasoning and intuition in a characteristic fashion. Mathematical insight is generally hard for students to develop. In order to develop spatial reasoning and model relations various approach has been developed as mentioned below.

Constructive Approach

Knowledge of shapes developed in all ancient cultures, involving spatial patterns and development of arts and culture.

Axiomatic Approach

An axiomatic system provides an explicit foundation for a mathematical subject. Axiomatic systems include six parts: 1. axioms (postulates, conjectures, common notions), 2. the logical language, 3. rules of proof or logical steps, 4. definitions (explanations), 5. propositions or theorems, and 6. proofs of theorems.

We know little about Euclid (circa 300 B.C.E.) except his mathematics, including the most influential mathematics book of all time: the Elements [13 Chapters]. In it he organized virtually all the elementary mathematics known at the time. The Elements contains definitions, axioms, and 465 theorems and their proofs—but no explanations or applications. For centuries the format represented the ideal for mathematicians and influenced many other areas of knowledge.

Although we live in a three-dimensional world, visualizing three-dimensional figures is harder than visualizing two-dimensional figures. Euclid devoted the final three books of his Elements to three-dimension.

Archimedes (287–212 B.C.E.) was another great mathematician of the ancient world and outstanding engineer and physicist.

Analytic Approach

The fruitful use of algebra to study shapes called analytic approach has become an indispensable tool for mathematicians, scientists, and those in many other fields. Although Rene Descartes ′ (1596–1650) and Pierre de Fermat (1601–1665) deserve credit for creating analytic approach, many others before and after shared in its development. And, many calculus texts include topics such as parametric equations and polar coordinates. In addition to these topics, later sections discuss Bezier curves in computer aided design and shapes in three and more dimensions.

We can realize that first and second degree equations correspond to lines and conics. We can solve some questions now considered part of calculus, such as finding maxima and minima of certain functions.

Surface Model Approach

The most common surfaces are spherical, elliptical and hyperbolic. The shortest path on the surface of a sphere connecting two points is a great circle – a circle with the same radius as the sphere. A circle is the intersection of the sphere and a plane that passes through the center of the sphere. Circles of longitude and the equator are great circles.

Transformational Approach

Moving geometric figures around is an ancient and natural approach. However, the Greek emphasis on logical or axiomatic approach and constructions and, later, the development of analytic approach overshadowed transformational thinking. Early in the twentieth century physicists realized the power of transformations, starting with Einstein's theory of relativity and then with quantum mechanics. We first investigate isometries – transformations preserving distance.

Members of the most important family of transformations, isometries, do not change the distance between points as the transformations take place or move them. Isometries are the dynamic counterpart of the notion of congruence.

Intermediate between isometries and affine transformations are similarities, the transformations corresponding to similar figures. The group of similarities includes more than the group of isometries, the transformations matching congruence. We define affine transformations by their matrix form.

Symmetry Study Approach

The rules of symmetry restrict how an artist can fit copies of a motif together to make a design. Archaeologists and anthropologists have started using symmetry in their study of designs to provide greater insight into cultures. Chemists and physicists use symmetry to organize new discoveries, to analyze empirical evidence, and to suggest fruitful lines for future inquiries.

The bodies of most animals illustrate bilateral symmetry; that is, a mirror reflection interchanges the two sides of the animal. Hunting lions as well as hunted antelopes need the ability to turn left as readily as right and to hear from each side equally well. However, feet are useful only underneath an animal, so there is no evolutionary advantage to a symmetry between up and down. Similarly, running backward isn't important for either predator or prey and we find no symmetry between the front and the back of animals. Hence the practical needs of most animals only require symmetry between right and left, but no other symmetry.

Human beings have used symmetry in art for thousands of years. Albrecht Durer (1471–1528) and others studied symmetry in art. Group theory and transformational geometry provided the mathematics needed to study symmetry. The symmetries of a figure are the transformations under which the figure is stable, and they always form a group.

The classification of symmetry groups has supplied scientists and others with a clear understanding of the patterns that can be found in their areas. Other insights about symmetry by twentieth century mathematicians, including H.S.M Coxeter, have affected many disciplines. Symmetric patterns, especially in physics, are often formal rather than visual. Even so, the same geometric figure intuition underlies symmetry. Questions from other disciplines have stretched the notion of symmetry and raised new mathematical questions. The beauty of the mathematics of symmetry and the beauty of symmetric objects have inspired the study of symmetry.

The beauty of crystals, especially gems, has fascinated people for centuries. However, prior to 1830 no chemical explanation existed for the regularities and other properties of crystals. The geometric classification by Hessel in 1830 of the 32 types of crystals spurred the study of geometric arrangements of atoms in crystals. Chemists sometimes say that there are 33 types; two are mirror reflections. The crystals are the three-dimensional analogs of frieze patterns and wallpaper patterns. A mathematical crystal is a discrete pattern having translations in at least three directions, not all in the same plane. The subgroup of translations takes a point to a three dimensional lattice of point.

Projective Approach

The progress made by the pioneers was overshadowed and mostly forgotten because of the marvelous advances of analytic geometry, vector, matrix and linear algebra and then calculus, which quickly dominated mathematical thought. Led by Gaspard Monge (1746–1818), the rediscovery and investigation of projective ideas started after 1800. Study of projections include computer graphics, statistical design theory, and photogrammetry (which infers geometric properties of objects based on photographs).

Finite Approach

Finite geometries as a subject arose from the investigations of geometric axioms at the end of the nineteenth century. The advent of hyperbolic and other geometries prompted a renewed Finite Geometries interest in axioms. In particular, geometers sought models satisfying certain axioms but not others. Often the models had finitely many points, whence the term finite geometries. The first explicitly finite geometry was a three-dimensional example with fifteen points developed in 1892 by Gino Fano. Geometers soon realized that finite geometries and their axiom systems were interesting for their own sake. Looking back, they found earlier mathematical designs, results, and problems anticipating finite geometry as a subject.

Current finite geometry research benefits from the cross-fertilization of geometry, algebra, and combinatorics in mathematics and areas outside of mathematics. Transformational geometry and group theory give powerful insights about finite geometries just as they have since the nineteenth century for traditional geometries. Combinatorics provides essential insights into finite and other areas of geometry. For example, the consequences of Euler's formula were identified through combinatorial reasoning. Error-correcting codes from coding Finite Geometries theory, are used in electronic data transmission and benefit from the interaction of these areas of mathematics.

Differential Approach

Intuitively, straight lines do not bend at all, while circles bend at constant rates depending on their radii.

In the 1730s Alexis-Claude Clairaut (1713–1765) and Leonhard Euler (1707–1783) considered space curves, such as the helix. They analyzed how much such curves twisted out of a plane, now called the torsion of a curve. More central to our focus, they and others investigated general surfaces and geodesics.

The inherent distortions of a flat map of portions of a curved earth motivated many mathematicians from Euler to Carl Friedrich Gauss (1777–1855). In fact, as part of his employment, Gauss spent some years surveying the state of Hanover, which spurred his profound investigations in differential geometry. His major paper in 1827 shaped the direction of this subject. Gauss studied a space on its own terms, not dependent on how it is embedded in flat space. For example, measurements of a portion of the surface of the earth could determine its curvature there. They link the area of a triangle in spherical and hyperbolic geometry to their angle sums. In the generalization Gauss used geodesics for the sides of the triangles. Then, as he proved, the difference between the angle sum of the triangle and π was determined by the area of the triangle and the curvature within it. The sphere has constant positive curvature and the hyperbolic plane has constant negative curvature.

In 1854 Georg Riemann (1826–1866) generalized Gauss's results to any number of dimensions and so transformed differential geometry again. Many mathematicians developed Riemann's insights into rigorous mathematics with computational methods. Albert Einstein (1879–1955) built on Riemann's profound insights to revolutionize physics with the general theory of relativity. Einstein needed a four-dimensional space built from three spatial dimensions and time, all with varying curvature corresponding to the strength of gravitational fields at different points.

Sir Issac Newton

In both mathematics and physics Sir Isaac Newton (1643–1727) stands as a giant in his own right, somewhere between Pope's intended epitaph and Newton's own words. At 18 Newton entered Cambridge University intending to study law. In his third year he started reading science and mathematics, including works of Galileo and Kepler. He realized he needed to know more geometry and set about reading a translation of Euclid's Elements. Soon after he studied Descartes' Geometry and other recent mathematics.

Newton received his bachelor's degree in 1665, just before the university had to close because of an outbreak of the plague in England. Newton went home and thought for two years, working out the essentials of mechanics in physics. He is famous for Laws of Motion, Universal Law of Gravity and work on Optics. Newton became the Lucasian professor at age 27.

An entire list of Newton's achievements in mathematics and physics would take far too much space. It is noteworthy that he used Euclidean geometric proofs rather than calculus in his foundational work in physics, the Principia Mathematica. He also was the first to use calculus to develop ideas now considered part of differential geometry.

Albert Einstein

His deep physical intuition helped make Albert Einstein (1879–1955) the greatest physicist since Newton. His undergraduate training was to be a mathematics and physics teacher, but he was unable to find a position when he graduated at age 22. He worked for several years in the Swiss patent office while he worked on his doctorate in physics, completing it in 1905. While still at the patent office he published groundbreaking papers on the special theory of relativity, quantum mechanics (photoelectric effect), and Brownian motion. The first two profoundly altered the study of physics, but in 1921 he received the Nobel prize in physics for the second. By this time he had published his general theory of relativity and was internationally known for his work in relativity and mostly for the formula $E=mc^2$.

Einstein became a professor in 1908 and after several posts in Switzerland, returned to Germany, where he was a professor until 1933. The rise of the Nazis caused him to renounce his German citizenship and emigrate to the United States. He was professor at Princeton and at the Institute for Advanced Study until his death. During World War II, his letter to President Franklin D. Roosevelt was instrumental in convincing Roosevelt to establish the research program resulting in the development of the atomic bomb. After the establishment of the state of Israel, he was offered the position of its first president, but declined the honor.

Einstein continued to work once he became world famous. However, neither he nor anyone since then has been able to achieve his goal of a unified field theory, which would combine quantum mechanics, relativity theory, and indeed all of physics. He doubted the standard interpretation of quantum mechanics with its inherent indeterminacy. In his efforts to counter it, he suggested a number of experiments that spurred others to important discoveries. However, all these discoveries supported the standard interpretation of quantum mechanics. While Einstein may not strictly be a mathematician, his theoretical physics depended strongly on geometry and mathematics in general. And mathematics and many mathematicians have been inspired by his profound insights.

Discrete Approach

Throughout the history of geometry mathematicians have explored a variety of questions and topics that are now considered part of discrete geometry. For example, what polygons can we use to cover the plane without gaps or overlaps? Can we always divide the interior of a polygon into triangular regions using only its vertices? If so, in how many ways? While polygons are traditional geometric objects, an answer to the questions won't be found in Euclid's text or high school geometry texts. For another example, how many different distances can a set of n points determine?

A number of mathematicians in the twentieth century focused on the preceding questions and similar ones that address combinatorial aspects of points, lines, polygons, and other configurations. Gradually the questions coalesced into a separate area of geometry, called discrete geometry. Part of the attraction of the field lies in how the solutions of two seemingly similar problems can range so widely in difficulty. The focus on problems and particularly the large number of unsolved problems reflect the newness of this field.

The increasing application of computers in many fields made some discrete geometry topics suddenly useful. For example, epidemiologists use Voronoi diagrams to study outbreaks of disease. However, there are often many possible sources, instead of the handful of pumps in Dr. Snow's map. With a large number of potential sources, scientists turn to computers to determine the Voronoi diagram and so look for clustering. Computational geometry arose in response to the need for efficient algorithms to implement a variety of geometric ideas using computers.

Math Principles

Universal Principle

The rules and relation described using mathematical language is universal meaning it is true for all types of physical quantities and measurement units.

Common Notions or Equality Principles

CN1. Reflexive Property $a = a$

$25 = 25$

CN2. Symmetric Property If $a = b$ then $b = a$

CN3. Transitive Property If $a = b$ and $b = c$, then $a = c$

$a+4 = 10$ and $10 = 6+4$, then $a+4 = 6+4$

CN4. Substitution Property

If $a = b$, then a can be used in place of b and vise versa.

If $a = 9$ and $a–c = 5$, then $9–c = 5$

CN5. Addition Property If $a = b$, then $a+c = b+c$.

If $2x = 6$, then $2x+5 = 6+5$

CN6. Subtraction Property If $a = b$, then $a–c = b–c$.

If $2x = 6$, then $2x-5 = 6-5$

CN7. Multiplication Property If $a = b$, then $ac = bc$.

If $y = 8$, then $5 \cdot y = 5 \cdot 8$

CN8. Division Property If $a = b$, then $a/c = b/c$.

If $3b = 18$, then $3b/3 = 18/3$

CN9. All Part Summation Property

If a, b, c, d are part of e then $a+b+c+d = e$

Geometrical Postulates

Six basic types of geometrical postulates has been categorized:

Existence Postulate

E1. There are at least minimum two points and unique straight line connecting them exist.

More explanation:

- Through any two points we can draw one, and only one, straight line.
- Two points A and B determine a unique straight line that contains them.
- Every line contains at least two points.

Distance Postulate

D1. Given points A and B, there is a unique real number, called the distance from A to B.

More explanation:

- If A and B are distinct points, there is a particularly useful coordinate function that assigns the coordinate 0 to x1 for A and a positive coordinate x2 for B, then distance AB or length of line AB is given by relation
  $AB = | x2 - x1 |$
- The distance from A to B satisfies the following properties:
  1. (Symmetry) $AB = BA$,
  2. (Non-negativity) $AB \geq 0$, and
  3. (Non-degeneracy) $AB = 0$ if and only if $A = B$.
- For every line there exists a coordinate function.

Plane Postulate

P1. Every plane contains at least three non-collinear points.

More explanation:

Given any three different non-collinear points, there is exactly one plane containing them.

Definition:

If A, B, and C are three points not on the same line, then the system of three segments AB, BC, CA, and their endpoints is called the triangle $\triangle ABC$. The three segments are called the sides of the triangle, and the three points are called the vertices.

If A, B and C are three points on the same line then according to the CN9 or All Part Summation Property of Equality

AC = AB + BC

P2. Space contains at least four non-coplanar points..

Order Postulate

O1. If point B is between points A and C, then A, B, C are distinct points on the same line, and B is between C and A.

O2. For any two distinct points A and C, there exist points B and D on the line AC with B between A and C and C between A and D.

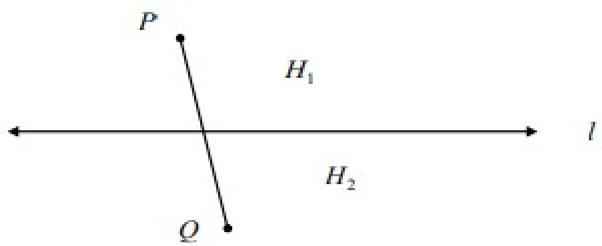
O3. If A, B, C are three distinct points on the same line, then one and only one of the points is between the other two.

Definition. By the segment AB is meant the set of all points that are between A and B. Points A and B are called the endpoints of the segment. The segment AB is the same as segment BA. Line is a collection or set of points.

Plane Separation Postulate

PS1. Give a line l, the points off l form two disjoint nonempty sets H1 and H2, called the half-planes bounded by l, with the following properties:

1. H1 and H2 are convex sets.
2. If P ∈ H1 and Q ∈ H2, then PQ cuts l



Two points A and B off line l are on the same side of l if they lie in the same half-plane bounded by l; otherwise A and B are on opposite sides of l.

More explanation:

A line l separates the points (of the plane) that are not on l into two sets such that if two points A and B are in the same set, the segment AB does not intersect l, and if A and B are in different sets, the segment AB does intersect m. In the first case A and B are said to be on the same side of m; in the second case, A and B are said to be on opposite sides of m. Actually there are only two sides to a line.

Definition:

By the ray AB is meant the set of points consisting of those that are between A and B, the point B itself, and all points C such that B is between A and C. The ray AB is said to emanate from point A. A point A, on a line m, divides m into two rays such that two points are on the same ray if and only if A is not between them.

Continuity Postulate

C1. (**Archimedes Continuity**) If AB and CD are two segments, there is some positive integer n such that n congruent copies of CD end to end from A along AB will pass beyond B.

C2. (**Dedekind's Continuit**y) For every partition of the points on a line into two nonempty sets such that no point of either lies between two points of the other, there is a point of one set which lies between every other point of that set and every point of the other set.

More explanation:

When a line passes from one side to the other side of a second line, it must share a point with the second line.

There are plenty of real numbers that aren't dyadic or rational length and there are plenty of points on l that aren't dyadic points. Dedekind's construction of real numbers, essentially says that you get to any point on a line if you take enough steps but there are no gaps in a line.

Dedekind's postulate is a sort of converse to the plane separation postulate. The essence of the postulate is that any point O on l separates all the other points on l into those to the left of O and those to the right. Dedekind's postulate says that, conversely, any separation of points on l into left and right is produced by a unique point O.

Archimedes postulate says that every straight line can be made to coincide with any other straight line, and this may be done so that an arbitrarily chosen point on the first coincides with any point chosen on the second. Two figures are called congruent if one can be put on top of other, such a way that all their parts coincide; in other words, two congruent figures are one and the same figure, in two different places.

Two equal segments AB, CD can be made to coincide in two different ways; namely, the point A falling on C and B on D, or A falling on D and B on C. In other words, one can turn the segment AB around in such a way that each of the points CD takes the place of the other.

There exist only three relations in terms of size with other object:

1. **Congruent or Equal measure**
2. **Greater than**
3. **Smaller than**

If any two above mentioned relation is false, logically it must be the third one.

Any figure congruent to a straight line is itself a straight line.

Properties of Line Segment Congruence

1. **Reflexive:** Every segment is congruent to itself
2. **Symmetric:** If AB is congruent to CD then CD is congruent to AB
3. **Transitive:** If AB is congruent to CD and CD to EF then AB is congruent to EF.

Triangle Geometry

Without preliminary knowledge of triangle we cannot understand Foundation of Geometry or circular understanding of geometry.

Angle

By definition of triangle we know a triangle has three vertices and three sides. However, triangle has three angles also since when two rays have the same endpoint, an angle is created. One remark may be about the way we defined the idea of angle. Under this simple definition, an angle is simply a set which is the union of two noncollinear lines with the same end-point.

Much later, in analytic geometry we need to talk about directed angles.

Angle Measurement

A protractor is a measuring device that measures how "open" an angle is. Angles are measured in degrees, and labeled with a $\circ$ symbol. Notice that there are two sets of measurements, one opening clockwise and one opening counter-clockwise, from 0° to 180°. When measuring angles, always line up one side with 0°, and see where the other side hits the protractor. The vertex lines up in the middle of the bottom line, where all the degree lines meet.

Angles can be classified, or grouped, into four different categories. A straight angle is when an angle measures 180°. A right angle is when an angle measures 90°. Acute angles are angles that measure between 0° and 90°. Obtuse angles are angles that measure between 90° and 180°. If two lines intersect to form right angles, the lines are perpendicular.

Congruent Angle

When two geometric figures have the same shape and size then they are congruent. An angle bisector is a line that divides an angle into two congruent angles, each having a measure exactly half of the original angle. Every angle has exactly one angle bisector.

Vertical angles are two non-adjacent angles formed by intersecting lines. Vertical angles are congruent to each other.

Angle Congruence has 3 properties:

1. **Reflexive**
2. **Symmetric**
3. **Transitive**

Linear Pair Angle

Adjacent angles are two angles that have the same vertex, share a side, and do not overlap.

A linear pair is two angles that are adjacent and whose non-common sides form a straight line. If two angles are a linear pair, then they are supplementary (their sum magnitude is equal to one straight angle or 180°).

Triangle Theorems

1. Triangle Sum Theorem: The sum of the interior angles in a triangle is one straight angle or 180°

2. Triangle Congruence

Recall that two figures are congruent if and only if they have exactly the same size and shape. If two triangles are congruent, they will have exactly the same three sides and exactly the same three angles. In other words, two triangles are congruent if you can turn, flip, and/or slide one so it fits exactly on the other. Hence congruent triangles have equal area in measure too.

Notice that when two triangles are congruent their three pairs of corresponding angles and their three pairs of corresponding sides are congruent. When referring to corresponding congruent parts of congruent triangles, you can use the phrase Corresponding Parts of Congruent Triangles are Congruent, or its abbreviation CPCTC.

Triangle congruence conditions:

- **SAS Congruence Condition:** If two sides and the included angle in one triangle are congruent to two sides and the included angle in another triangle, then the two triangles are congruent.
- **ASA Congruence Condition:** If two angles and a non-included side in one triangle are congruent to two corresponding angles and a non-included side in another triangle, then the triangles are congruent.

- **SSS Congruence Condition:** The SSS Triangle Congruence Condition states that if three sides in one triangle are congruent to three sides in another triangle, then the triangles are congruent.

3. Triangle Perimeter

Perimeter is the distance around a shape. The perimeter of any figure must have a unit of measurement attached to it.

Hence perimeter of Triangle is actually the sum of its three length or **$P = a+b+c$**

4. Triangle Area

Area is the amount of planar space covered by a figure. Area is measured in square units.

The area of any triangle is equal to half the base times the height. This method applies no matter which side is considered the base and perpendicular height to the base from noncollinear vertex point.

<u>Area SSS or Heron's Formula</u>

$$\sqrt{s(s-a)(s-b)(s-c)} \text{ where } s = (a+b+c) / 2$$

5. Right Triangle – Pythagorean Theorem and Ratio

<u>Pythagorean Theorem</u>

The two shorter sides of a right triangle (the sides that form the right angle) are the legs and the longer side is the hypotenuse. The Pythagorean Theorem states that **$a^2 + b^2 = c^2$** where the legs are "a" and "b" and the hypotenuse is "c". Another name given is The Sulva Sutra simply. However various name has been attributed also like Katyayana, Bodhayana, and Apastamba Sulva Sutra.

We can extend the converse of the Pythagorean Theorem to determine if a triangle has an obtuse angle or is acute. We know that if the sum of the squares of the two smaller sides equals the square of the larger side, then the triangle is right. We can also interpret the outcome if the sum of the squares of the smaller sides does not equal the square of the third.

- If the sum of the squares of the two shorter sides in a right triangle is greater than the square of the longest side, then the triangle is acute.
- If the sum of the squares of the two shorter sides in a right triangle is less than the square of the longest side, then the triangle is obtuse.

In other words: The sides of a triangle are a, b and c and c > b and c > a

If $\mathbf{a^2 + b^2 > c^2}$, then the triangle is acute.
If $\mathbf{a^2 + b^2 = c^2}$, then the triangle is right.
If $\mathbf{a^2 + b^2 < c^2}$, then the triangle is obtuse.

Ratio

The three basic ratios called sine, cosine and tangent can be defined. At this point, we will take the sine only.

Sine Ratio: For an acute angle θ in a right triangle, the $\sin\theta$ is defined as the ratio of the side opposite the angle over the hypotenuse of the triangle and its value is given by the following formula

$$(27900\,\theta\,(180-\theta)+\theta^2(180-\theta)^2)\,/\,291600000$$

or

$$4\,\theta\,(180-\theta)\,/\,(40500-\theta(180-\theta))$$

For higher precision value you need knowledge of pi.

6. Law of Sine

In any triangle ABC with angles α, β, γ and sides a, b, c one can write

$$\sin\alpha\,/\,a=\sin\beta\,/\,b=\sin\gamma\,/\,c$$

This is a relationship between the interior angles of a triangle and the sides opposite these angles. This relationship is known as the law of sine.

4

Culture, Math and History

Numbers – One to Four

Let us look at the first four primary numbers in the spirit of culture. The number **ONE** can of course define a quantity; as, for example, one apple. But in its other sense, it perfectly represents the principle of absolute unity, and as such has often been used as the symbol to represent God. As a statement of form it can in one sense represent a point – it has been called the 'pointal' number, the bindu or seed in the mandala – or in another sense it can represent the perfect circle. **TWO** is a quantity, but symbolically it represents the principle of Duality, the power of multiplicity. At the same time it has its formal sense in the representation of a line, in that two points define a line. **THREE** is a quantity, but as a principle it represents the Trinity. Its formal sense is that of the triangle, which is formed from three points. With three a qualitative transition is made from the pure, abstract elements of point and line to the tangible, measurable state which is called a surface. In Tantra the triangle is called the Mother, for it is the membrane or birth channel through which all the transcendent powers of unity and its initial division into polarity must pass in order to enter into the manifest realm of surface. The triangle acts as the mother of form. But three is yet only a principle of creation, forming the passage between the transcendent and the manifest realms, whereas **FOUR** represents at last the 'first born material thing', the world of Nature, because it is the product of the procreative process, that is of multiplication: $2 \times 2 = 4$. As a form, four is the square, and represents materialization.

The universality of Number can be seen in another, more physical context. We learn from modern physics that from gravity to electromagnetism, light, heat, and even in what we think of as solid matter itself, the entire perceptible universe is composed of vibrations, perceived by us as wave phenomena. Waves are pure temporal patterns, that is dynamic configurations composed of amplitude, interval and frequency, and they can be defined and understood by us only through Number. Thus our whole universe is reducible to Number. Every living body physically vibrates, all elemental or inanimate matter vibrates molecularly or atomically, and every vibrating body emits a sound. The study of sound, as the ancients intuited, provides a key to the understanding of the universe.

Music is pervaded by the fundamental law of reciprocity; changes in frequency and wave length are reciprocal. Rising or falling tones, as reciprocal arithmetic ratios, are applied to string-lengths. 'Major' and 'minor' are reciprocal tonal patterns. Plato conceived the World-Soul as constituted of reciprocal ratios identical with those which, in Agamic mythology, create the musical 'Drum of Shiva', the pulsating instrument of creation.

Whether the product of an eastern or a western culture, the circular mandala or sacred diagram is a familiar and pervasive image throughout the history of art. Nepal, Tibet, India, Islam and medieval Europe have all produced them in abundance, and most tribal cultures employ them as well, either in the form of paintings or buildings or dances. Such diagrams are often based on the division of the circle into four quarters, and all the parts and elements involved are interrelated into a unified design. They are most often in some way cosmological; that is, they represent in symbol what is thought to be the essential structure of the universe: for example, the four spatial directions, the four elements, the four seasons, various divinities and often man himself. But what is most consistently striking about this form of diagram is that it expresses the notion of cosmos, that is of reality conceived as an organized, unified whole.

One of the most striking uses of the mandala is dome architecture. The square represents the earth held in fourfold embrace by the circular vault of the sky and hence subject to the ever-flowing wheel of time. When the incessant movement of the universe, depicted by the circle, yields to comprehensible order, one finds the square. The square then presupposes the circle and results from it. The relationship of form and movement, space and time, is evoked by the mandala.

The Sri Yantra is drawn from nine triangles, four pointed downward and five pointed upward, thus forming 42 (6×7) triangular fragments around a central triangle. There is probably no other set of triangles which interlock with such integrational perfection.

Zero and One Principle

I would like to consider in some detail two symbol, **Zero** and **One**, because they provide an exceptional example of how mathematical concepts are the prototypes for the dynamics of thought, **of action** and **of structuring**.

We know now that we exist in groups, determined by various levels of energetic affinities, repelling, exchanging and absorbing through interconnected, subtle energetic communications. And our being extends outward through various energy fields to connect with larger fields. We have had to learn that there is nowhere that we can dispose of the things we have finished using – that there is no zero drain in our sink; there is no factory pipe or hole in the ground that does not lead somewhere. Everything remains here with us; the cycles of growth, utilization and decay are unbroken. With zero we have at the beginning of modern mathematics a number concept which is philosophically misleading and one which creates a separation between our system of numerical symbols and the structure of the natural world. On the other hand, the notion of Unity remains, literally, unthinkable; simply because in order for anything to be, to exist, it must, in the very positive affirmation of itself, negate that which it is not. Cold is only cold because it is the negation of heat. For a thing to be, its opposite must also be. Everybody knows it is symbolically represented in **Yin-Yang Principle** or **Shiva-Shakti Principle**.

The origin and concept of the symbol **zero** or **0** as a place value begins with **ABACUS** the first calculating device. In Yogic doctrines we can find its expression and zero is accepted as reality. This school laid exclusive emphasis on the goal of obtaining personal transcendence and escape from karma through renunciation of the natural world. The goal of this highly ascetic pursuit was the attainment of an utterly impersonal reality, a total cessation of movement within I-ego centric consciousness. A description of it attributed to Nagarjuna is 'a state of incognizable, imperishable, selfless absence'. This single aspect or possibility of meditative experience was held to be the ultimate goal of the created Universe as well as the goal of all individual spiritual development. Rich spiritual heritage of Yoga, tradition which upholds a spiritual significance in both the manifested and the unmanifested expressions of God, and whose tantric and yogic practices work towards an intensification of the relationship and harmonization between matter and spirit. From this time that the concept of zero took on a new tangibility and presence. The name given to this concept is sunya. Hence zero provided a framework for the development of atheism or negation of absolute reality of material world.

In India after Brahmagupta the zero is treated as a tangible entity, as a number. Aristotle and other Greek teachers had talked about the concept of zero philosophically, but Greek mathematics, fortified as it was by the Pythagorean teaching from Egypt, resisted the incorporation of zero into its system. Maya civilization used a sort of egg-shaped symbol.

The idea of the unknowable Unity at the beginning has been the basis of many philosophies and mythological systems. According to Shankharacharya the main stream of Vedanta has always rested on the notion of the One, the Divine, who divided himself within himself to form his own self-created opposite, the manifested universe. Within the divine self-regard, three qualities of himself became distinguished: Sat (immobile being), Chit (consciousness-force) and Ananda (bliss). The original unity, represented by a circle, is then restated in the concept of the Real Idea, the thought of God, called the bindu or seed, what we call the geometrical point.

The point, according to the Shiva Sutra Vimarshini Commentaries, forms the limit between the manifest and non-manifest, between the spatial and the non-spatial. The bindu corresponds to the 'seed-sound idea' of the Tantras. The Divine transforms himself into sound vibration (nada), and proliferates the universe, which is not different from himself, by giving form or verbal expression to this self-idea. Thus the universe springs forth from the Word.

This transcendent Word is only a vibration (a materialization) of the Divine thought which gives rise to the fractioning of unity which is creation. The Word (sabda in Nepali, the logos of the Christians and Gnostics), whose nature is pure vibration, represents the essential nature of all that exists. Concentric vibrational waves span outward from innumerable centers and their overlappings (interference patterns) form nodules (generic term derived from node) of trapped energy which become the whirling, fiery bodies of the heavens. The Real-Idea, the Purusha, the inaudible and invisible point of the sound-idea remains fixed and immutable. Its names, however, can be investigated through geometry and number.

This emitted sound, the naming of God's idea, is what the Pythagorean would call the Music of the Spheres. In ancient Egypt the primordial vibrational field (called nada) is called Nun, the primal ocean. It is the One imaged as undifferentiated cosmic substance, the source of all creation. Submerged within this primal ocean is Atum, the creator, who must first distinguish himself from Nun in order for creation to begin. Atum is masculine, and analogous to Chit (consciousness-force) of the Yogic mythology. Atum is pictured in a state of total self-absorbed bliss.

Thus the most recent scientific model of creation is allied to the image given in ancient mythology, and both acknowledge an absolute singularity or Unity at the beginning. In terms of the orthodoxy of ancient mathematics, the symbols of mathematics should reflect the realities they describe. With zero and the army of merely mental and statistical signs which followed from it, we are not very far from having a system of mathematical symbols which corresponds to the pure order of Nature.

The word Nature which is also called prakrti means 'that which is born', and all birth into nature requires this crossing of opposites. So the square came to represent the earth, and as such symbolized the conscious experience of finite existence, of what is born into Nature. This brings us to the problem of whether the sides of the square are curved or straight: if the overall reality of the universe is an endless curvature, an endless movement, there is yet a consciousness which is capable of temporarily arresting, both conceptually and perceptually, segments of the universal continuum. This objective consciousness might be seen as a reduced velocity of the universal consciousness, and has as its instrument the cerebral cortex in man.

Various terms like Ajā, Avyakta, etc., are used as synonyms of prakrti. Prakrti is the matrix of the universe, the material cause of all. It is called prakrti, since it procreates the entire world. It is called Pradhāna because everything ultimately dissolves into it. It is called Avyakta, as it cannot be perceived by ordinary sense organs. It is Ajā, since it is the root-cause of all and is not the product of anything else.

History of Numbers

Evidence suggests that the exact beginning of mathematical thought, even the origin of the concept of a "number," predates the advent of written language. The earliest example of recorded mathematical symbols is a sequence of tally marks on the leg bone of a baboon found in Swaziland, dating to around 35,000 years ago. By contrast, the earliest known language writings date to around 6,500 years ago. It is not known for sure what the tally marks on the bone represent, but it is plausible that they represent a record of an early hunter's kills. These tally marks may represent numbers in application, but there may also be evidence that early humans were interested in properties of numbers themselves.

Possible evidence of mathematics more sophisticated than counting comes from another bone dated ten thousand years younger than the Swaziland counting bone. Around 25,000 years ago, by the shores of Lake Edward (which today lies on the border between Uganda and Zaire), the Ishango people lived in a small fishing, hunting, and farming community. This settlement lasted for a few centuries before being buried in a volcanic eruption.

Jump 10,000 years forward and a bit to the east from the makers of the Ishango bone, and you're in the emerging Egyptian and Fertile Crescent civilizations. These civilizations had deep understanding of mathematics and used it to achieve unequalled engineering feats. Babylonian clay tablets show an understanding of Pythagorean triples, centuries before the cult of Pythagoras appeared in Greece. However, these were not the only ancient civilizations to develop, presumably independently, familiarity with numbers and number relationships. Mathematical concepts may have spread naturally throughout Africa, and the Middle East, and Asia, but they also appeared early on in Central and South America.

Evidence suggests that the development of mathematics in early cultures was tied to specific purposes. Much of early mathematical thought was focused on representing and understanding the movements of the heavens, as is evident in the Mayan long count calendar and the constellations of the zodiac. Elsewhere, such as in ancient China, mathematics was put to use in bookkeeping and other business activities. Math's development generally served practical purposes until about 600 BC, when the Greek philosophers began to explore the world of numbers itself.

Negative Numbers

Students might feel that the concept of numbers less than zero just doesn't make sense. They wouldn't be alone. There have been several classifications of numbers, including negative numbers, that were not been well received initially. Negative numbers have existed in various forms for a couple of thousand years, but didn't really catch on in Europe until the 1500. The Greek mathematician Diophantus (250 AD) considered equations that yielded numbers less than zero to be "absurd." So, it is understandable if students today take a while to warm up to them. It is true that numbers less than zero do not make sense in every context, but there are plenty of places where they do.

Numbers less than zero is used in various areas: Temperature (temperatures below zero), golf scores (shots under par), money (being in debt), bank accounts (being overdrawn), negative hyperbolic curvature, elevation (being below sea level), years (AD vs BC), latitudes south of the equator, the rate of inflation (falling prices), electricity (impedance or voltage can be negative), the stock market (the Dow was down 47 points), and bad Jeopardy scores. launches of rockets run through the number line. T minus 3 means it's 3 seconds before the test. T minus 3, T minus 2, T minus 1, 0, 1, 2... counts the seconds before and after lift off: -3, -2, -1, 0, 1, 2...

Those real life examples can be used to make sense of some of the rules for computation. The fact that $4 + (-9) = (-5)$ can make sense to students because they know that dropping 9 degrees from a temperature of 4 degrees makes it 5 degrees below zero. Also, if it is currently 17 degrees, but the temperature is dropping 5 degrees a day for the next 7 days, that is the same assaying that $17 + 7(-5) = -18$. That same situation can show that two negatives multiplied together make a positive. Again, suppose this day it is 17 degrees and the temperature is dropping 5 degrees a day. going back in time, four days ago it must have been $17 + (-4)(-5) = 37$ degrees.

The use of vector mathematics and its physical reality mostly in dynamics and kinematics, provides sufficient understanding of negative principles and quantity as a negative number. The high degree reality of vector mathematics in nature is found in plant physiology process.

There is another number concept something called complex number system. The physical working reality of this number system is mostly found in electromagnetism.

Modern development of quantum physics also assertion the negativity of number.

5

Music and Mathematics

Sri Yantra

The rishis of the Rig Veda probably originated from the flood myth, constructing large harmonic mountains such as for 432,000 "years" for the Kali Yuga within a Maha Yuga system of 4,320,000 "years" since the universal flood-hero number is 8,640,000,000. The vedic scale involved twenty-two srutis from which the seven scales naturally emerge under the different limit of 4,320 and 8,640, three times larger than 1,440 and 2,880.

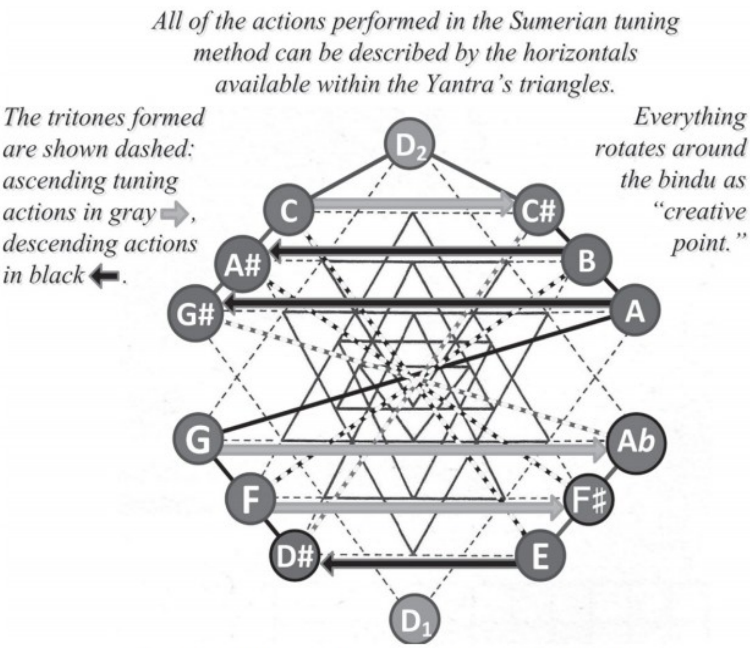
After the survial, they created Sri Yantra one of the greatest yantra of the Mahavidhya or Nigamanatmic Tantric School dedicated to the goddess Lalita. So music is called Lalitkala also. It is evolved from a bindu point into a series of interlocking upward-and downward-facing isosceles triangles, with an axis of symmetry like the tone circle.

As usual, one needs to tune a lyre in the Pythagorean manner using fifths and fourths, working in three pairs of symmetrical tunings from the octave limits, that is string 1 and string 8. In theory this forms the Pythagorean heptatonic, equivalent in scale to the modern Dorian, though the final semitone gaps of E–F and B–C might be adjusted by ear to sharpen the leimma by something close to the syntonic comma of 81/80.

The modern Dorian would be the Sumerian embubu tuning, containing the tritone B to F. This allows either side of the tritone to dissolve its adjacent semitone and move the scale order to a new mode. A new semitone is formed at one end from an existing tone, then still presenting a new tritone between two semitones, ready to repeat the process but then from the opposite side of the tritone. The relationship to the Sri Yantra comes from the fact that each of six inner note classes are only modified, by a semitone, once in the entire process, and there are six vertically symmetrical horizontal vertices of the triangles that make up the yantra. We can now move to the inner significance given to tuning in Vedic Culture. The modern world is founded on objects and objective facts: we collect them, prove them, catalogue them, and even count ourselves among them. This impulse to be something is, however, but one side of reality, since nothing comes to be what it is without some interval within which it came to be. Were one to focus on such intervals to study them rather than their fruits (which are tones as vibrations) then one might master becoming rather than be a collector of facts.

This perspective has something to say to us about the purpose of learning in the modern world: that we are over-identified with the results of our own actions and unwilling to sacrifice immediate gratifications for either skill development or an investigation into the purpose of our working. So it is in the area of harmony: there is chaos until one turns to how tones arise through intervals. The notion of sacrifice applied as strongly to understanding how one understands as it did to understanding the harmonic world. Objects and objective facts are the preferred worldview for sight rather than sound.

Sound in motion is the very illumination and institution of this [Rig Vedic] vision . . . intent on becoming that which it knows: the creator, the means of the creation, and the creation.



The inner world of the sensorium is where meaning can happen, but within it lies a set of rules that hides control of the meaning-making process. The thrust of the Rig Veda is to transform consciousness through dis-identification, where identification is the creation of a god on the mountain. Limiting numbers are the limits of what can happen within them, and what can happen is a form of consciousness, a god: as in Genesis, **"You shall be as gods and know good and evil."**

Consciousness does not exist without questioning reality, and such questioning is suppressed when it is wholly identified with something. There has to be an unknowing for consciousness to arise, and I believe it was in this sense the writers of the Rig Veda proposed their system of inner sacrifices, of what they thought. There exists overall view that harmony is not a fixed absolute but a continuing and dynamic relativity. And when we look at human beings there is no capacity for being able to understand everything, nor any need to. All of human life is a relativity, yet absolutes such as ideologies can capture the human relativity and turn it into a monster of identification.

And if humans are relative systems, and we know harmony is a relative system (having only one fixed reference tone in its body), there are only intervals. So what are intervals or modal scales or octaves? Harmony is a massive system of interrelation possible through number and without any necessary physical concerns (such as friction) until music needs to be made on an earthly instrument. To realize harmony there must be vibrations in the air on earth while, in heaven, planets roll around in orbits that do not alter over millions of years, hence vibrating through an eternal motion relative to each other. This is not to say that there are not absolute invariants but that these invariants are then applied relatively within real systems.

If harmony is subjective, it can be objectified as a material fact while also being part of human nature through the senses and part of the human mind through number. And it is highly likely that the human mind includes that part that can receive harmony, perhaps because the heart (not the physical heart but a spiritual center) is the true organ of knowing.

The mind, strongly linked to language, is an instrument that can play itself, and the god in this mind is identification, over which many minds have no control. When a particular mental identification is broken, for whatever reason, something new arises that was hidden behind the identification. While this fact is in principle well-known, the power of the identification process must, like a "dragon," be consciously defeated before the "gold" on which it "sleeps" can enter the human ecology of mind—yet society as a whole is found powerless to see the problem as due to their own thinking, treating gods, myths, and cosmological doctrines as absolute rather than transitional objects.

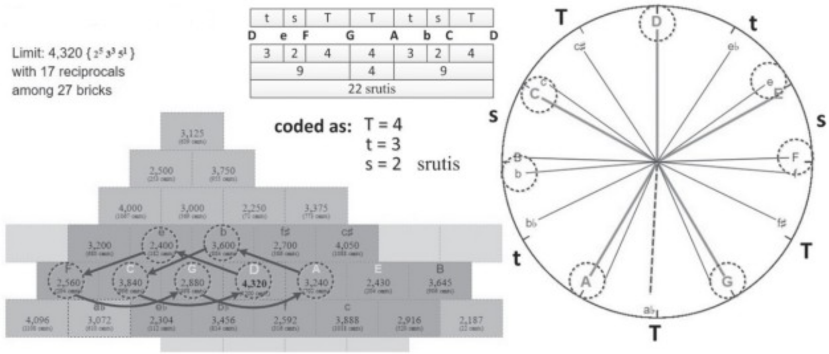
Something about the world holds humanity in identification, leading to destructive unhappiness and lack of fulfillment. And so, the harmonic doctrine, in relating to human development, appears to have moved (since 600 BCE) to ideas of self-transformation, liberation, nirvana, self-sacrifice, avatars, saviors, and messiahs. And the harmonists of the Rig Veda seem early to have laid out how they would use the gods to do their creative work with consciousness, a far more relevant aim than following the fate determined by the gods thought possessed by the divine. In this way, the subjective nature of harmony, when established, could speak to the subjective human nature and its purpose on Earth.

The connection of the human mind with cosmology—at whatever viewpoint-defining level, everyone always has one—is a relativity that seems to orient the sense of identity within the world. They presented systematic discipline and approach to change consciousness, defeat identifications, and not remain trapped by them. Their model for this was associated with the need to sacrifice what is already thought, and today, the human condition regularly inflicts its process model of the world on itself as an object identity.

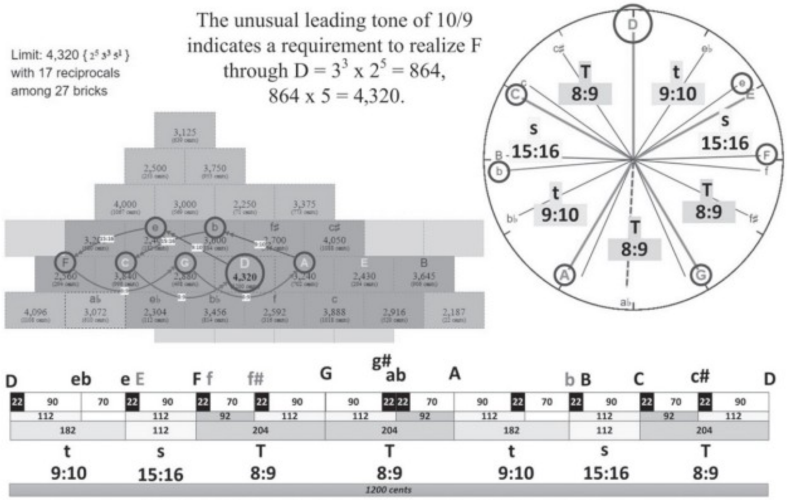
The rishis who wrote the Rig Veda saw their own makeup as being harmonic within their inner harmonic worldview, based on music however Vedic scales differed from Greek or modern scales in their use of seven syllables to indicate steps, that is, intervals, rather than notes. The vedic rishis came to a division into twenty-two subintervals rather than seven intervals within an octave scale or modern twelve semitones. However, the need for twenty-two srutis (as they are called) hinges on the ability to ring all the possible changes within the octave, given a just intonation taken to a very high limiting number. Yet, while the numeracy implied might be high, the twenty-two srutis were a masterful simplification of all the potentials possible to the harmonic mountains that interested the rishis and their expression of just intonation.

In the following figure, showing the srutis, a just diatonic scale is primarily involved, and secondarily, the Pythagorean F is used that, in tuning order F–C–G–D, requires a limit for D with a harmonic root of three cubed (33 = 27) so as to enable three steps equaling a descending interval of 16/27 to reach F. It is also therefore true that, in order to reach F on the mountain, using practical srutis, the limiting number has to be 4,320 in order to manifest the reciprocal tone B. This number 4,320 is at the heart of Vedic and post-Vedic large numbers, involving the creator god Brahma and the yuga cycle. And the scale is therefore asymmetrical relative to the G–A axis of symmetry (wherein the fundamental tritone of Agni resides, in-between).

While srutis are normally vaguely defined, a fixed system was originally intended to be made up of three srutis: syntonic commas; chromatic semitones; and the Pythagorean leimma.

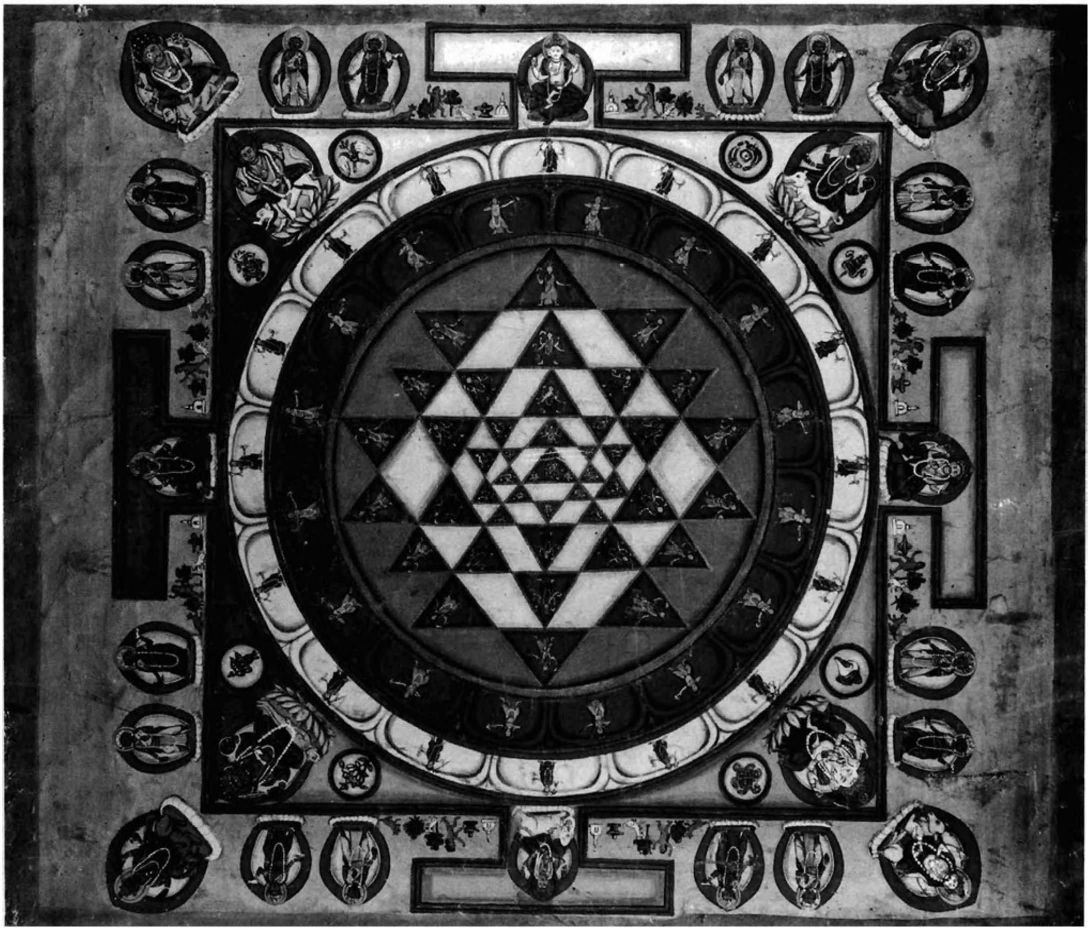


The limit 4,320, necessary to generate Vedic division of the octave. The location for D belongs to the 4,320:8,640 numbers of Vedic sacred time periods. It has extra width in its central register F–C–G–D–A–E–B, which is heptatonic and offers multiple versions of the same just scales on its mountain. The scale marked on the mountain (left) is also marked on the tone circle (right), within which one can see the srutis making up tones 10/9 and 9/8 and semitone 16/15. Yet there are Pythagorean leimmas, syntonic commas, and chromatic semitones dividing these up. Unrolling the tone circle, one can clearly see how every interval within the scale is made up of two or three srutis, with values of 22, 70, and 90 cents. This indicates how the Bible and the Vedas chose different but adjacent locations for D on the mountains they developed.



How this looks in the context of the whole octave can be realized by the limit 4,320, which is a just-tuning version of the Pythagorean heptatonic scalelimit of 864 (864 × 5 = 4,320). Doubling 4,320 to 8,640 creates a fuller rhomboid and hence enables all seven just scales, available to limit 2,880, while also providing the fully Pythagorean scale (F–C–G–D–A– E–B).

The super-gods of India, such as Brahma Prajapati and Indra and the system of yugas, seem to have the same limit of 4,320 or (× 2) 8,640 or (× 2) 17,280 times a number of powers of 10 to define them.



Mathematical Analysis of Music

One of the most fundamental ways that music and math are connected is in the understanding of sound specifically, and wave phenomena in general. Understanding sound as an instance of wave phenomena provides a nice forum for the interaction of ideas from music, physics, and mathematics. Tools that have been developed to help us understand the nature of sound, such as Fourier analysis, can be generalized to shed light on many areas of mathematics. In return, the mathematical understanding of sound has helped foster the development of new technologies that extend the possibilities for musical exploration.

To the mathematician, wave structure and theory open the door to the examination of periodic functions, some of the most basic forms of patterns in mathematics.

Music, as an academic subject, ascended to a special place in the learned classes of those cultures that would carry on their intellectual traditions. For example, the "Quadrivium," composed of music, arithmetic, geometry, and astronomy, represented the curriculum of classical education for centuries. Such was the perceived value of musical education in the classical world that it was made one of the four core subjects. However, the musical studies of the Quadrivium focused mainly on the Pythagorean notion of ratios and scales rather than on the performance of musical compositions. Students learned about harmonics and the proportions that would yield pleasing scales and melodies. This focus on the structure of music is closer to what, in the modern age, would be called "music theory."

This was to be a mere prelude to the understandings that future mathematicians would bring to music. One of the most powerful connections to be discovered was that music, and sound in general, travels in waves. The mathematics of sound, of waves, to which we will now turn our attention, will lead us to powerful ways of thinking not only about music, but about many other phenomena.

In general, strings of different lengths produce sound of different frequencies. Without considering such things as string thickness or tension, longer strings tend to produce lower frequencies than do shorter strings. So, when two strings of different lengths are plucked together, the resulting sound is a combination of frequencies. A string vibrates with some fundamental frequency, 440 Hz for an "A" note, for example, but there are other frequencies present as well. These are known as either partials or overtones, and they give each instrument its characteristic sound, or timbre. Timbre helps explain why a tuba sounds different than a cello, even though you can play a "middle C" on both instruments.

For a single plucked string, the overtones occur at frequencies that are whole number multiples of the fundamental frequency. So, a string vibrating at 440 Hz (an "A") will also have some vibration at 880 Hz (440 × 2), 1320 Hz (440 × 3), and so on. These additional frequencies have smaller amplitudes than does the fundamental frequency and are, thus, more noticeable as added texture in a sound rather than as altered pitch.

Every instrument has its own timbre. If you play a middle A, corresponding to 440 Hz, on a piano, the note will have a much different sound than the same note played on a trumpet. This is due to the fact that, although both notes are based on the fundamental frequency of 440 Hz, they have different combinations of overtones attributable to the unique makeup of each instrument. If you've ever heard "harmonics" played on a guitar, you have some sense of how a tone can be made of different parts. When a guitarist plays "harmonics," he or she dampens a string at a very precise spot corresponding to some fraction of the string's length, thereby effectively muting the fundamental frequency of the vibrating string. The only sounds remaining are the overtones, which sound "thinner" than the fundamental tones and almost ethereal.

Up until this point, the connections we have drawn between music and math have been mainly physical, with a few somewhat philosophical ideas thrown in as well.

Fourier said that any function can be represented mathematically as a combination of basic periodic functions, sine waves and cosine waves. To create any complicated function, one need only add together basic waves of differing frequency, amplitude, and phase. In music, this means that we can theoretically make any tone of any timbre if we know which waves to use and in which relative amounts to use them. It's not unlike making a meal from a recipe—you need a list of ingredients, you need to know how much of each ingredient to use, and you need to know how and in what order to combine them.

The ingredients used in Fourier analysis are simply sine and cosine waves. Of course, these simple waves can come in different frequencies. For sounds that we consider pleasing and musical, the sine wave mostly will come in frequencies that are whole number multiples of a fundamental frequency. For sounds that are "noisy," such as white noise, the sine-wave ingredient frequencies can be anything.

In addition to helping us to distinguish the sounds of music, Fourier analysis has broad application in many other fields, as well. Its signal-processing capabilities are of use to scientists studying earthquakes, electronics, wireless communication, and a whole host of other applications. Any field that involves looking at or using signals to convey information, which covers a pretty broad swath of modern endeavors in science and business, uses Fourier analysis in some way or another.

Cause of Sound:

- Sound is caused by compression and rarefaction of air molecules.
- We perceive the amplitude of a sound wave as its loudness, or volume.
- We perceive the frequency of a sound wave as its pitch.
- An instrument's tone, the sound it produces, is a complex mixture of waves of different frequencies.
- Instruments produce notes that have a fundamental frequency in combination with multiples of that frequency known as partials or overtones.
- You can add manual spaces and text as well as building blocks to an entry. Manual spaces are inelegant, but can sometimes be a workaround to the one-tab limit.

Periodic Functions of Sound:

- Trigonometric functions, such as sine and cosine, are useful for modeling sound waves, because they oscillate between values.
- We can connect the idea of the sine function of an angle to sine waves dependent on time by analyzing the

"spoke" of a unit circle as it rotates, forming the hypotenuse of various right triangles.

- A sine wave can represent a sound wave theoretically, but not pictorially.
- The shape of a sine wave is altogether different than the "shape" of a sound wave found in nature.
- The wave equation uses second derivatives to relate acceleration in space to acceleration in time.
- We can think of the combination of sine waves in a Fourier series as a cooking recipe in which the full wave is a combination of varying amounts (amplitudes) of waves of various frequencies.
- To build a sawtooth wave out of sine waves, we need to know which frequencies and amplitudes to use.
- Fourier's chief contribution was a method for determining which amplitudes, frequencies, and phases of the trigonometric functions are needed to model any function.
- The Fourier series representation of the sawtooth wave is an infinite sum of sine waves.
- After a function has been converted into a Fourier series representation, it can be viewed in the frequency domain as opposed to the time domain.
- Frequency domain views provide a different, and sometimes more enlightening, perspective on the behavior of signals.

Music and Emotion

Music works because it connects to emotions. And it does so, among others, by reminding us of the sounds (and emotions connected to them) that we experienced before birth. Percussion instruments remind us of the heart beat of our mother and ourselves, cord and wind instruments remind us of all the voices we heard back then. Musical instruments are especially beautiful if they are driven and modulated by the body and the art of the player. All classical instruments are optimized to allow this modulation and the expression of emotions. The connection between the musician and the instrument is most intense for the human voice; the next approximation are the wind instruments.

Every musical instrument, the human voice included, consists of four elements: an energy source, an oscillating sound source, one or more resonators, and a radiating surface or orifice. In the human voice, the energy source is formed by the muscles of the thorax and belly, the sound source are vocal folds – also called vocal cords – the resonator is the vocal tract, and the mouth and nose form the orifice.

The breath of the singer or of the wind instrument player provides the energy for the sound generation and gives the input for the feedback loop that sets the pitch. While singing, the air passes the vocal folds. The rapid air flow reduces the air pressure, which attracts the cords to each other and thus reduces the cross section for the air flow. This pressure reduction is described by the Bernoulli equation. As a result of the smaller cross section, the airflow is reduced, the pressure rises again, and the vocal cords open up again. This leads to larger airflow, and the circle starts again. The change between larger and smaller cord distance repeats so that sound is produced; the sound is then amplified in the mouth by the resonances that depend on the shape of the oral cavity. Using modern vocabulary, singing a steady note is a specific case of self organization.

But how can a small instrument like the vocal tract achieve sounds more intense than that of a trombone, which is several meters long when unwound? How can the voice cover a range of 80 dB in intensity? How can the voice achieve up to five, even eight octaves in fundamental frequency with just two vocal folds? And how can the human voice produce its unmatched variation in timbre? Many details of these questions are still subject of research, though the general connections are now known.

The human vocal folds are on average the size of a thumb's nail; but they can vary in length and tension. The vocal folds have three components. Above all, they contain a ligament that can sustain large changes in tension or stress and forms the basic structure; such a ligament is needed to achieve a wide range of frequencies. Secondly, 90 % of the vocal folds is made of muscles, so that the stress and thus the frequency range can be increased even further. Finally, the cords are covered by a mucosa, a fluid-containing membrane that is optimized to enter in oscillation, through surface waves, when air passes by. This strongly non-linear system achieves, in exceptional singers, up to 5 octaves of fundamental pitch range.

Also the resonators of the human voice are exceptional. Despite the small size available, the non-linear properties of the resonators in the vocal tract – especially the effect called inertive reactance – allow to produce high intensity sound. This complex system, together with intense training, produces the frequencies, timbres and musical sequences that we enjoy in operas, in jazz, and in all other vocal performances. In fact, several results from research into the human voice – which were also deduced with the help of magnetic resonance imaging – are now regularly used to train and teach singers, in particular on when to use open mouth and when to use closed-mouth singing, or when to lower the larynx.

Singing is thus beautiful also because it is a non-linear effect. In fact, all instruments are non-linear oscillators. In reed instruments, such as the clarinet, the reed has the role of the vocal cords, and the pipe has the role of the resonator, and the mechanisms shift the opening that has the role of the mouth and lips. In brass instruments, such as the trombone, the lips play the role of the reed. In airflow instruments, such as the flute, the feedback loop is due to another effect: at the sound-producing edge, the airflow is deflected by the sound itself.

The second reason that music is beautiful is due to the way the frequencies of the notes are selected. Certain frequencies sound agreeable to the ear when they are played together or closely after each other; other produce a sense of tension. Already the ancient Greek had discovered that these sensations depend exclusively on the ratio of the frequencies, or as musician say, on the interval between the pitches.

More specifically, a frequency ratio of 2 – musicians call the interval an octave – is the most agreeable consonance. A ratio of 3/2 (called perfect fifth) is the next most agreeable, followed by the ratio 4/3 (a perfect fourth), the ratio 5/4 (a major third) and the ratio 6/5 (a [third, minor]minor third). The choice of the first third in a scale has an important effect on the average emotions expressed by the music and is therefore also taken over in the name of the scale. Songs in C major generally have a more happy tune, whereas songs in A minor tend to sound sadder.

The least agreeable frequency ratios, the dissonances, are the tritone (7/5, also called augmented fourth or diminished fifth or false quint) and, to a lesser extent, the major and minor seventh (15/8 and 9/5). The tritone is used for the siren in German red cross vans. Long sequences of dissonances have the effect to induce trance; they are common in Balinese music and in jazz.

After centuries of experimenting, these results lead to a standardized arrangement of the notes and their frequencies that is shown in the following figure. The arrangement, called the equal intonation or well-tempered intonation, contains approximations to all the mentioned intervals; the approximations have the advantage that they allow the transposition of music to lower or higher notes. This is not possible with the ideal, so-called just intonation.

Equal-tempered frequency ratio	1	1.059	1.122	1.189	1.260	1.335	1.414	1.498	1.587	1.682	1.782	1.888	2
Just intonation frequency ratio	1		9/8	6/5	5/4	4/3	none	3/2	8/5	5/3		15/8	2
Appears as harmonic nr.	1,2,4,8		9		5, 10			3, 6			c. 7	15	1,2,4,8

6

Science and Nature

Modern View

In science today the fundamental nature of the material world is knowable only through its underlying patterns of wave forms. The point of view of modern force-field theory and wave mechanics corresponds to the ancient geometric-harmonic vision of universal order as being an interwoven configuration of wave patterns.

In biology, the fundamental role of geometry and proportion becomes even more evident when we consider that moment by moment, year by year, every atom of every molecule of both living and inorganic substance is being changed and replaced. Every one of us within the next five to seven years will have a completely new body, down to the very last atom. Amid this constancy of change, where can we find the basis for all that which appears to be consistent and stable? Biologically we may look to our ideas of genetic coding as the vehicle of replication and continuity, but this coding does not lie in the particular atoms (or carbon, hydrogen, oxygen and nitrogen) of which the gene substance, DNA, is composed; these are all also subject to continual change and replacement. Thus the carrier of continuity is not only the molecular composition of the DNA, but also its helix form. This form is responsible for the replicating power of the DNA. The helix, which is a type from the group of regular spirals, results from sets of fixed geometric proportions. These proportions can be understood to exist apriori, without any material counterpart, as abstract, geometric relationships. The architecture of bodily existence is determined by an invisible, immaterial world of pure form and geometry.

Modern biology increasingly recognizes the importance of the form and the bonding relationships of the few substances which comprise the molecular body of living organisms. Plants, for example, can carry out the process of photosynthesis only because the carbon, hydrogen, nitrogen and magnesium of the chlorophyll molecule are arranged in a complex twelve fold symmetrical pattern. It seems that the same constituents in any other arrangement cannot transform the radiant energy of light into life substance. The specialization of cells in the body's tissue is determined in part by the spatial position of each cell in relation to other cells in its region, as well as by an informational image of the totality to which it belongs. This spatial awareness on a cellular level may be thought of as the innate geometry of life. All our sense organs function in response to the geometrical or proportional – not quantitative – differences inherent in the stimuli they receive. For example, when we smell a rose we are not responding to the chemical substances of its perfume, but instead to the geometry of their molecular construction. That is to say, any chemical substance that is bonded together in the same geometry as that of the rose will smell as sweet. Similarly, we do not hear simple quantitative differences in sound wave frequencies, but rather the logarithmic, proportional differences between frequencies, logarithmic expansion being the basis of the geometry of spirals.

Our visual sense differs from our sense of touch only because the nerves of the retina are not tuned to the same range of frequencies as are the nerves embedded in our skin. If our tactile or haptic sensibilities were responsive to the same frequencies as our eyes, then all material objects would be perceived to be as ethereal as projections of light and shadow. Our different perceptual faculties such as sight, hearing, touch and smell are a result then of various proportioned reductions of one vast spectrum of vibratory frequencies. We can understand these proportional relationships as a sort of geometry of perception. Within the human consciousness is the unique ability to perceive the transparency between absolute, permanent relationships, contained in the insubstantial forms of a geometric order, and the transitory, changing forms of our actual world. The content of our experience results from an immaterial, abstract, geometric architecture which is composed of harmonic waves of energy, nodes of relationality, melodic forms springing forth from the eternal realm of geometric proportion.

Colour and Quantum Theory

Quantum theory explains all colours in nature. Indeed, all the colours that we observe are due to electrically charged particles. More precisely, colours are due to the interactions of charged particles with photons. All colours are thus quantum effects.

The charged particles at the basis of most colours are electrons and nuclei, including their composites, from ions, atoms and molecules to fluids and solids. Many colour issues are still topic of research.

Some Colour Type	Cause
Wood fire, candle	Wood and wax flames are yellow due to incandescence if carbon-rich and oxygen-poor
Bluish water, blue ice when clear, violet iodine, red-brown bromine, yellow-green chlorine, red flames from CN or blue-green flames from CH, some gas lasers, blue ozone leading to blue and grey evening sky	Colours are due to quantized levels of rotation and vibrations in molecules
Ruby, emerald, alexandrite, perovskites, corresponding lasers	Electronic states of transition metal ions are excited by light and thus absorb specific wavelengths
Carbon arc lamp, hot steel, light bulb wire, most stars, magma, lava, hot melts	Incandescence and free charge radiation Colours are due to continuous spectrum emitted by all hot matter; colour sequence, given by Wien's rule

Some Colour Type	Cause
Gold (green in transmission), pyrite, iron, brass, alloys, silver, copper, ruby glass	Colours in reflection and in transmission are due to transitions of electrons between overlapping bands
Red haemoglobin in blood, blue blood haemocyanin, green chlorophyll in plants, yellow or orange carotenes in carrots, flowers and yellow autumn leaves, red or purple anthocyanins inberries, flowers and red autumn leaves	Colours are due to conjugated π-bonds, i.e. to alternating single and double bonds in molecules; floral pigments are almost all anthocyanins, betalains or carotenes
Blue sapphire, blue lapislazuli, green amazonite, brown-black magnetite $Fe_3O_4$and most other iron minerals	Light induces change of position of an electron from one atom to another; for example, in blue sapphire the transition is between Ti and Fe impurities; many paint pigments use charge transfer colours; fluorescent analytical reagents are used in molecular medicine and biology

Sea water, like fresh water, is blue because it absorbs red and green light. The absorption is due to a vibrational band of the water molecule that is due to a combination of symmetric and asymmetric molecular stretches. The absorption is weak, but noticeable. At 700nm (red), the $1/e$ absorption length of water is 1m. Sea water can also be of bright colour if the sea floor reflects light. In addition, sea water can be green, if it contains small particles that scatter or absorb blue light. Most often, these particles are soil or plankton. The sea is especially blue if it is deep, quiet and cold; in that case, the ground is distant, soil is not mixed into the water, and the plankton content is low. Lakes can also be blue if they contain small mineral particles. The particles scatter light and lead to a blue colour for reasons similar to the blue colour of the sky. Such blue lakes are found in many places on Earth.

Colours fascinate. Fascination always also means business; indeed, a large part of the chemical industry is dedicated to synthesizing colourants for paints, inks, clothes, food and cosmetics. Also evolution uses the fascination of colours for its own business, namely propagating life. The specialists in this domain are the flowering plants. The chemistry of colour production in plants is extremely involved and at least as interesting as the production of colours in factories. Practically all flower colourants, from white, yellow, orange, red to blue, are from three chemical classes: the carotenoids, the anthocyanins (flavonoids) and the betalains. These colourants are stored in petals inside dedicated containers, the vacuoles.

Even though colours are common in plants and animals, most higher animals do not produce many colourants themselves. For example, humans produce only one colourant: melanin. (Hemoglobin, which colours blood red, is not a dedicated colourant, but transports the oxygen from the lungs through the body. Also the pink myoglobin in the muscles is not a dedicated colourant.) Many higher animals, such as birds, need to eat the colourants that are so characteristic for their appearance. The yellow colour of legs of pigeons is an example. It has been shown that the connection between colour and nutrition is regularly used by potential mates to judge from the body colours whether a proposing partner is sufficiently healthy, and thus sufficiently attractive.

The study of the colours of incandescence, led Max Planck to discover the quantum of action. In the meantime, research has confirmed that in each class, all colours are due to the quantum of action $\hbar$. The relation between the quantum of action and the material properties of atoms, molecules, liquids and solids are so well known that colourants can now be designed on the computer. An exploration of the causes of colours found in nature confirms that all colours are due to quantum effects.

Understanding the colour lines produced by each element had started to become interesting already before the discovery of helium; but afterwards the interest increased further, thanks to the increasing number of applications of colour knowledge in chemistry, physics, technology, crystallography, biology and lasers.

Colours are specific mixtures of light frequencies. Light is an electromagnetic wave and is emitted by moving charges. For a physicist, colours thus result from the interaction of charged matter with the electromagnetic field. Now, sharp colour lines cannot be explained by classical electrodynamics. We need modern quantum theory to explain them.

Simply saying, the quantum of action $\hbar$– together with the interaction between electromagnetic fields and the electrons inside atoms, molecules, liquids and solids – determines the size, the shape, the colour and the material properties of all things around us. The quantum of action determines mechanical properties such as hardness or elasticity, magnetic properties, thermal properties such as heat capacity or heat of condensation, optical properties such as transparency, and electrical properties such as metallic shine. In addition, the quantum of action determines all chemical and biological aspects of matter. The strength of the electromagnetic interaction is described by the fine-structure constant $\alpha \approx 1/137.036$.